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LINEAR STABILITY ANALYSIS OF CORAL BED

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ABSTRACT

Coral is symbiotic with algae called zooxanthella, and grows by nutrients produced by photosynthesis of zooxanthella. It is assumed that a channel bed is composed of some kind of coral. In this study, linear stability analysis of coral bed is conducted by using three governing equations; Exner equation, Momentum equation, Continuity equation and the stability of coral bed is analyzed for the conditions without and with considering Reynolds stress. The governing equations are normalized to perform linear stability analysis. From the analysis, it is found that in the unstable region (without Reynold's stress), the growth rate increases monotonically with increasing wave number. For the condition (with Reynold's stress), it is revealed that the growth rate vanishes when the wave number goes to infinity. There is found a dominant wave number associated with a maximum growth rate. From the growth rate vs. Froude number curve, it is seen that the growth number is minimum when the Froude number is negative or the flow pattern is subcritical. On the other hand, the growth rate becomes neutral when the flow pattern is supercritical. From the growth rate vs wave speed curve, it is observed that the growth rate decreases with the increase of wave speed, and in a certain position, it becomes neutral.

Keywords: *Coral bed, Linear stability, Froude number, Wave number, Reynolds stress.*

1. INTRODUCTION

Coral is symbiotic with algae called zooxanthella and grows by nutrients produced by photosynthesis of zooxanthella. These beds (Figure 1) are formed of colonies of coral polyps held together by calcium carbonate. Most of the beds are built from stony corals, whose polyps cluster in groups. The coral bed serves ecosystem services for tourism, fisheries, and shoreline protection (Nagelkerken et al. 2002).

Figure 1: image of Coral bed

In fluid dynamics, the ambition of determining linear stability is to observe if a given flow is either stable or unstable, and if so, how these instabilities will affect the progress of turbulence is verified. The determination of linear stability is most particularly laid by Helmholtz, Kelvin, Rayleigh, and Reynolds. In this research, the linear stability of the coral bed will be defined, both respecting Reynold's stress and without considering Reynolds stress. The coral bed is very significant for many different motives aside from hypothetically comprising the most assorted ecosystem on the sphere. It defends coastlines from the harmful effects of wave conflicts and tropical storms and delivers locales and shelter for many marine creatures. It is the source of nitrogen and other vital nutrients for marine food chains supports in carbon and nitrogen setting and help with nutrient recycling.

In prior Linear stability has been analyzed in growth system of sand bars in anabranching rivers (Z.W, Li et al. 2013), meander development initiating from alternate bars (Fujita et al. 1985), gravel-bed river driven by vegetation (Y.Shimizu et al. 2013), river channel flows (Callandar, R.A et al. 1969), closed conduit (Hosoda et al. 2000) and so on. But linear stability analysis of coral bed is rarely found. As linear stability analysis is a very crucial function of fluid dynamics, research on the analysis of linear stability in coral beds has been shown in this paper.

It is supposed that necessary substances for photosynthesis such as carbon dioxide are abundant, and, as a result, photosynthesis is mainly controlled by light. In addition, coral is eroded if the flow velocity is higher than some critical velocity. Assuming that the availability of light decreases inversely proportionally to the flow depth and erosion rate is proportional to the flow velocity subtracted by the critical velocity. A simple sketch of a coral bed is shown in (Figure 2).

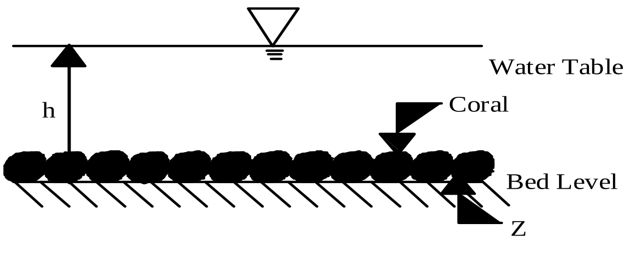

Figure 2: Simple sketch of coral bed.

In mathematics, in the concept of differential equations, if the resolution has the form, where A is a linear operative whose spectrum comprises eigenvalues with a positive real part. If all the eigenvalues have a negative real part, the solution is termed as linearly stable. An operating point of the system model is said to be linearly stable if, when perturbed from an operating point by a slight amount, the system model returns to that operating point (Sudhoff et al. 2003).

In this research, the three governing equations; Exner equation, Momentum equation, Continuity equation are normalized to make linear stability analysis, and the stability of coral bed is also analyzed for the conditions without and with considering Reynold's stress

2. METHODOLOGY

In this section, Exner equation, momentum equation, continuity equation, boundary condition, normalization of the governing equation, and asymptotic expansions are discussed.

2.1 Governing Equations

The governing equations Exner equation, Momentum equation and continuity equation are presented below:
Exner equation,

$$\frac{\partial \tilde{Z}}{\partial \tilde{t}} = \frac{\tilde{\alpha}}{\tilde{H}} - \tilde{\beta}(\tilde{U} - \tilde{U}_c) \tag{1}$$

Where $\tilde{Z}$ is the bed elevation, $\tilde{t}$ is time, $\tilde{\alpha}$ and $\tilde{\beta}$ are empirical constants with some dimensions, $\tilde{U}$ and $\tilde{H}$ is the flow velocity and depth respectively, $\tilde{U}_c$ is the critical flow velocity for the erosion of coral, and the tide denotes dimensional variables and are later removed to denote normalized equivalents

Flow in the channel is assumed to be described by Momentum equation for two conditions.

Without Reynold's stress:

$$\tilde{U}\frac{\partial \tilde{U}}{\partial \tilde{x}} = -g\frac{\partial \tilde{H}}{\partial \tilde{x}} - g\frac{\partial \tilde{Z}}{\partial \tilde{x}} + gS - \frac{\tilde{\tau}_b}{\rho \tilde{H}} \tag{2}$$

With Reynold stress:

$$\tilde{U}\frac{\partial \tilde{U}}{\partial \tilde{x}} = -g\frac{\partial \tilde{H}}{\partial \tilde{x}} - g\frac{\partial \tilde{Z}}{\partial \tilde{x}} + gS - \frac{\tilde{\tau}_b}{\rho \tilde{H}} - \tilde{\varepsilon}\frac{\partial^2 \tilde{U}}{\partial \tilde{x}^2} \tag{3}$$

Where $\tilde{x}$ is the streamwise coordinate, $\tilde{\tau}_b$ is the bed stress, ρ is the water density, g is the gravity acceleration, S is the bed slope, $\tilde{Z}$ is the bed elevation from the flat bed with a constant slope S, Reynold's stress $\tilde{\varepsilon} = 0.077\tilde{U}_f \tilde{H}_n$.

Where, $\tilde{\tau}_b = \rho C_f \tilde{U}^2$. (4)

Here, C_f is the friction coefficient which is assumed to be a constant for simplicity.

The Continuity equation is

$$\tilde{U}\tilde{H} = \tilde{q} \tag{5}$$

Where, $\tilde{q}$ is the net discharge. It is assumed that the growth rate of coral is balanced with the erosion rate, the equilibrium state is realized. Thus the bed remains flat with its constant elevation. The diagram of linear stability analysis with variables are shown in Figure 3, where, *Z* is the bed level, *U* and *H* are the flow velocity and depth respectively (dimensional).

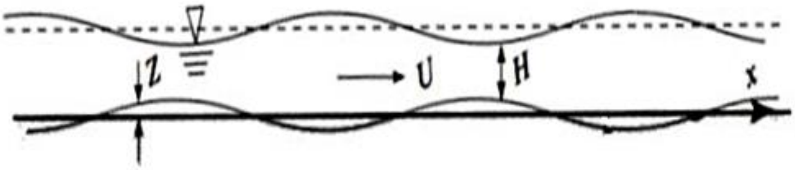

Figure 3: Diagram of linear stability analysis

2.2 Normalization

The following normalization has been used for normalization of the Exner equation, Momentum equation and continuity equation.

The equations (1), (2), (3), (4), (5) are normalized using the following equations (6), (7), (8), (9)

$$(\tilde{U},\tilde{U}_c) = \tilde{U}_n(U,U_c) \quad (6)$$

$$(\tilde{x},\tilde{H},\tilde{Z}) = \tilde{H}_n(x,H,Z) \quad (7)$$

$$\tilde{t} = \frac{\tilde{H}_n{}^2}{\tilde{\alpha}} t \quad (8)$$

$$\tilde{\beta} = \frac{\tilde{\alpha}}{\tilde{q}} \beta \quad (9)$$

The normalized form of Exner equation.

$$\frac{\partial Z}{\partial t} - \frac{1}{H} + \beta(U - U_c) = 0 \quad (10)$$

The normalized form of Momentum equation

$$U\frac{\partial U}{\partial x} + F^{-2}\left(\frac{\partial H}{\partial x} + \frac{\partial Z}{\partial x}\right) - C_f\left(1 - \frac{U^2}{H}\right) = 0 \quad (11)$$

$$U\frac{\partial U}{\partial x} + F^{-2}\left(\frac{\partial H}{\partial x} + \frac{\partial Z}{\partial x}\right) - C_f\left(1 - \frac{U^2}{H}\right) + \varepsilon F^{-1}\frac{\partial^2 U}{\partial x^2} = 0 \quad (12)$$

Equation (6) and (7) are the normalized form of momentum equation without and with the Reynold's stress respectively.
The normalized form of Continuity equation.

$$UH - 1 = 0 \quad (13)$$

2.3 Linear Stability Analysis

For performing linear stability analysis, we have employed the asymptotic expansions of the sinusoidal form as,

$$u = A\hat{u}\exp[ik(x - \omega t)] \quad (14)$$

$$h = A\hat{h}\exp[ik(x - \omega t)] \quad (15)$$

$$z = A\hat{z}\exp[ik(x - \omega t)] \quad (16)$$

Where, A, k and ω are the amplitude, wavenumber and complex growth rate of perturbation ($\omega = \omega_r + i\omega_i$), respectively.

By performing linear stability analysis the following equations are obtained,

$$\omega = \left(F^{-2}\frac{U_c}{1-U_c}\right)\frac{3C_f + ik(1-F^{-2})}{9C_f{}^2 + k^2(1-F^{-2})^2} \quad (17)$$

From equation (17) separating the real and imaginary part we find,

$$\omega_r = \left(F^{-2}\frac{U_c}{1-U_c}\right)\frac{3C_f}{9C_f{}^2 + k^2(1-F^{-2})^2} \quad (18)$$

$$\omega_i = \left(F^{-2}\frac{U_c}{1-U_c}\right)\frac{k(1-F^{-2})}{9C_f{}^2 + k^2(1-F^{-2})^2} \quad (19)$$

After Linear stability analysis the complex growth rate of perturbation ($\omega = \omega_r + i\omega_i$) with considering Reynold's stress is goven below.

$$\omega = (F^{-2} \frac{U_c}{1-U_c}) \frac{3C_f + ik(1-F^{-2}+\frac{k^2\varepsilon}{F})}{9{C_f}^2 + k^2(1-F^{-2}+\frac{k^2\varepsilon}{F})^2} \tag{19}$$

After separating the real and imaginary part from equation (19) it is perceieved that,

$$\omega_r = (F^{-2} \frac{U_c}{1-U_c}) \frac{3C_f}{9{C_f}^2 + k^2(1-F^{-2}+\frac{k^2\varepsilon}{F})^2} \tag{20}$$

$$\omega_i = (F^{-2} \frac{U_c}{1-U_c}) \frac{k(1-F^{-2}+\frac{k^2\varepsilon}{F})}{9{C_f}^2 + k^2(1-F^{-2}+\frac{k^2\varepsilon}{F})^2} \tag{21}$$

Here, $\omega_i =$ growth rate of perturbation.

3. RESULT AND DISCUSSIONS

To analyze the linear stability of coral bed, the variation of Froude number and wave number for both Reynold's stress and without Reynold's stress 'MATHMATICA 9' software has been used.

3.1 Instability diagram

Instability diagram without considering Reynold's stress and considering Reynold's stress is shown below.

3.1.1 For Without Reynold's Stress

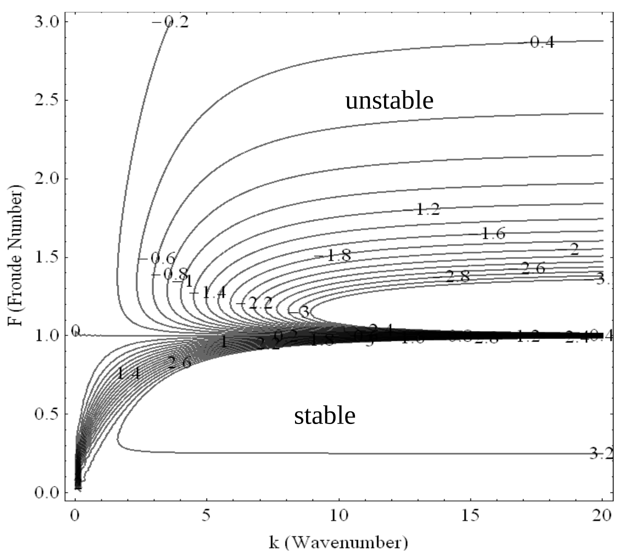

Figure 4: The contours of growth rate of perturbation (without Reynold's stress)

Figure 4 shows the contours of the growth rate of perturbation in the case for without Reynolds stress (variation of Froude number with wave number). In figure 4, ω_i is negative when $F \geq 1$. From Figure 4, it can be said that the flat bed is unstable when the Froude number is larger than unity. Because ω_r is always positive, the bed waves migrate downstream. The imaginary part of the angular frequency ω_i corresponds to the growth rate of perturbation so that the flat bed becomes unstable when *F* is greater than unity. In the unstable region, the growth rate ω_i increases monotonically with increasing wave number *k*, as *k* goes to infinity the growth rate asymptotically approaches to a constant value gradually.

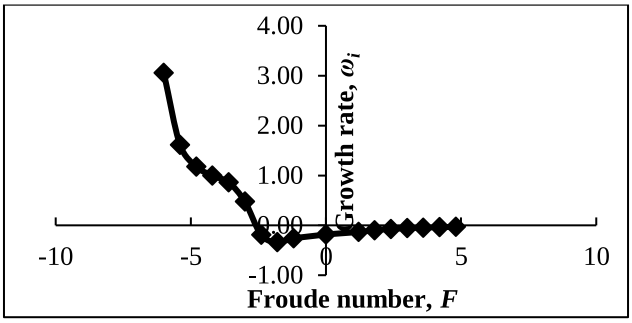

Figure 5: Variation of growth rate with Froude number (without Reynold's stress)

Here, in Figure 5, Growth rate ω_i is gradually decreasing with the increasing Froude number *F*. Growth rate is minimum when Froude number is approximately (-2). After that the growth rate starts to increase up to Froude number is approximately 5. Then the growth rate gradually becomes neutral with the increase of Froude number.

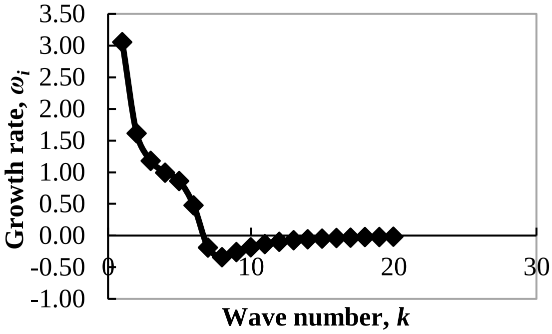

Figure 6: Variation of growth rate with wave number (without Reynold's stress)

Figure 6 indicates that, Growth rate ω_i is gradually decreasing with the increasing wave number *k*. Growth rate is minimum when wave number is approximately 7. After that growth rate starts to increase up to growth rate is approximately 15. Then the growth rate gradually becomes neutral with the increase of wave number.

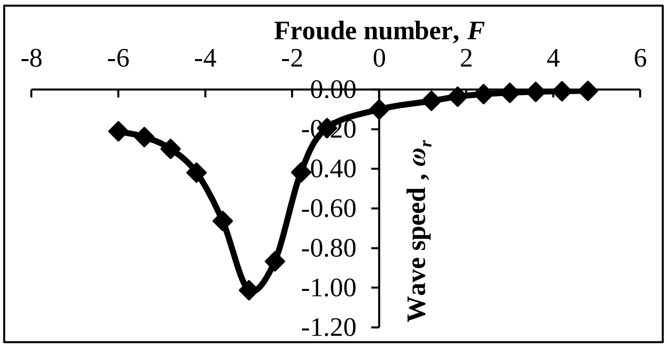

Figure 7: Variation of wave speed with Froude number (without Reynold's stress)

It is observed in Figure 7, wave speed ω_r is gradually decreasing with the increasing Froude number *F*. Wave speed is minimum Froude number is approximately (-3). After that wave speed starts to increase up to Froude number is approximately (+3). Then the wave speed gradually becomes neutral with the increase of wave number.

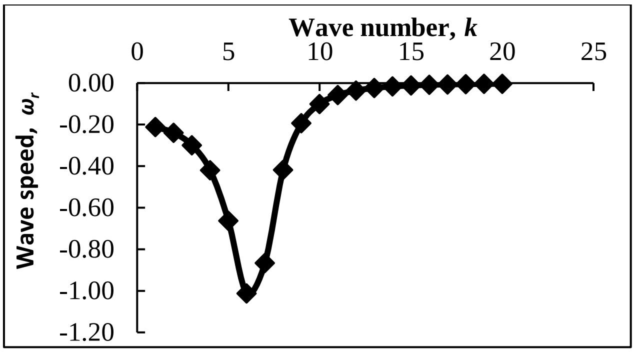

Figure 8: Variation of wave speed with wave number (without Reynold's stress)

Here, in Figure 8, wave speed ω_r is gradually decreasing with the increasing wave number *k*. Wave speed is minimum when wave number is approximately 6. After that wave speed starts to increase up to wave number is approximately 15. Then the wave speed gradually becomes neutral with the increase of wave number.

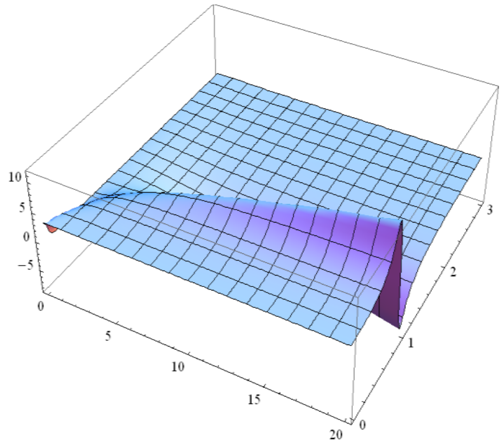

Figure 9: 3-D view of growth rate of perturbation (without Reynold's stress)

3.1.2 For Reynold's Stress

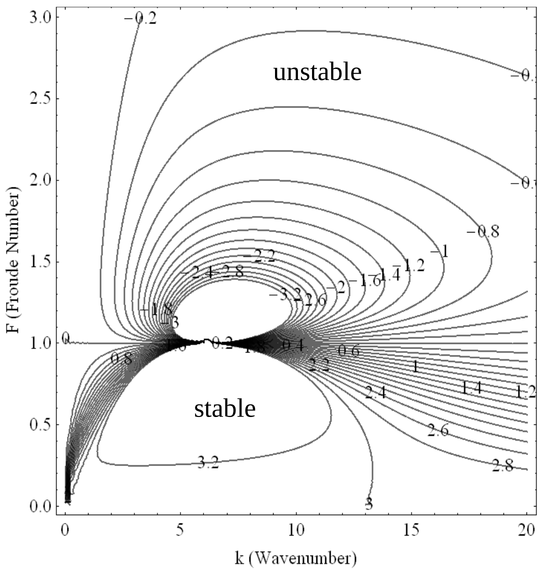

Figure 10: The contours of growth rate of perturbation (with Reynold's stress)

Therefore, from this Figure 10, it is found that ω_i is negative when $F \geq 1$ and the bed is unstable. In the unstable region the growth rate ω_i increases with increasing wave number, k in the small range of k. As, k goes to infinity, however, ω_i vanishes in this case because of the additional term originated from the Reynolds stress. There is a dominant wave number associated with a maximum growth rate.

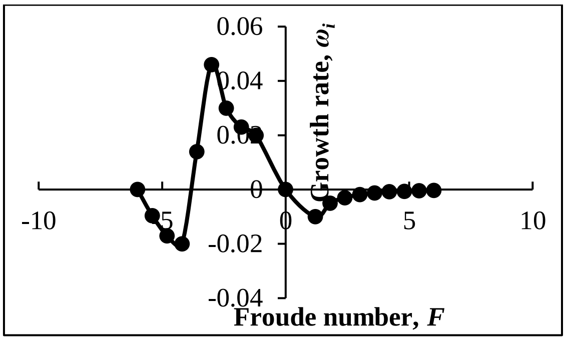

Figure 11: Variation of growth rate with Froude number (with Reynold's stress)

In this Figure 11, growth rate, ω_i is decreasing with the increase of Froude number, F. Growth rate decreases up to Froude number is approximately (-4). Then it starts to increase up to Froude number is approximately (-2.5). Then the growth rate again starts to decrease. When the Froude number is approximately 4 the growth rate becomes neutral gradually.

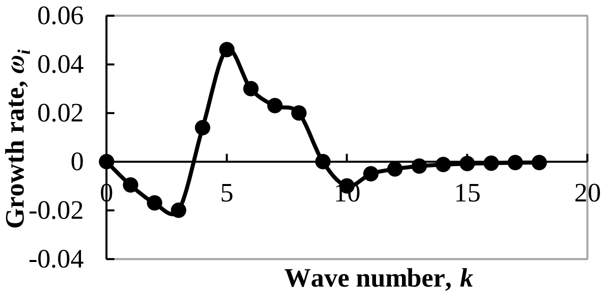

Figure 12: Variation of growth rate with wave number (with Reynold's stress)

In this Figure 12 growth rate, ω_i is decreasing with the increase of wave number, k. Growth rate decreases up to wave number is approximately 4. Then it starts to increase up to wave number is approximately 5. Then the growth rate again starts to decrease. When the wave number is approximately 15 the growth rate becomes neutral gradually.

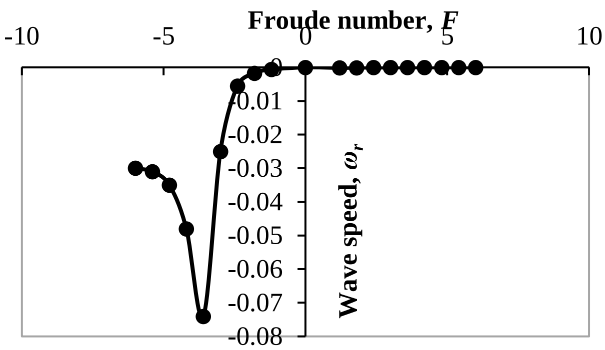

Figure 13: Variation of wave speed with Froude number (with Reynold's stress)

In Figure 13, wave speed, ω_r is decreasing with the increase of Froude number. The wave speed decreases up to Froude number is approximately (-4). Then it starts to increase up to the Froude number is approximately (-1) and then the wave speed becomes neutral gradually.

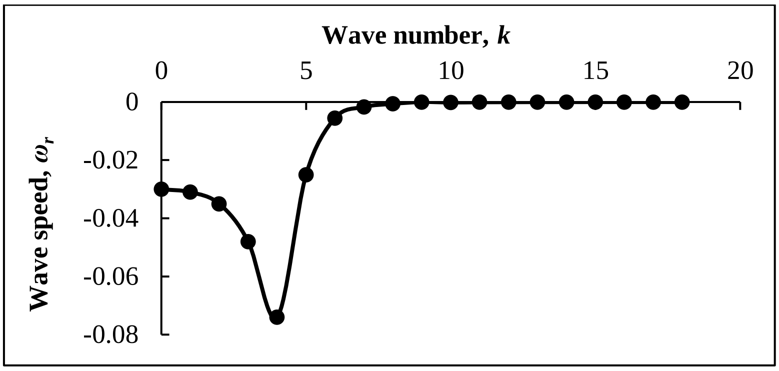

Figure 14: Variation of wave speed with wave number (with Reynold's stress)

From the above figure it is observed, wave speed, ω_r is decreasing with the increase of wave number. The wave speed decreases up to wave number is approximately 4. Then it starts to increase up to the wave number is approximately 7 and then the wave speed becomes neutral gradually.

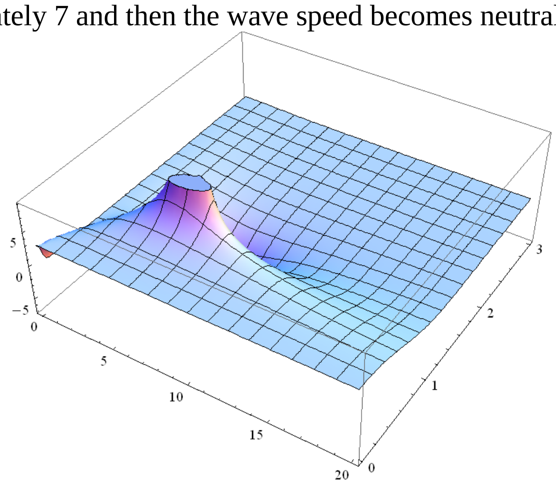

Figure 15: 3-D view of growth rate of perturbation (with Reynold's stress)

4. CONCLUSIONS

In this study, linear stability analysis of coral bed is conducted. Exner equation, Momentum equation, Continuity equation are used. Perturbation technique has been used to analyze the equations for stability analysis. From this analysis some conclusions are made. They are, stability of coral bed depends on Froude number and Wave number. Unstable region in the instability diagram expands in the direction of increasing wave numbers. Growth rate is negative and bed is unstable when Froude number is greater than unity. When wave number goes to infinity growth rate approaches to a constant position when Reynolds stress is absent. Growth rate vanishes when wave number goes to infinity when Reynold's stress is present.

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